Yiddish word(s) of the day (ייִדיש וואָרט פֿון טאָג)

to be happy = freylekh = פֿריילעך
tsufridn = צופֿרידן
gliklekh = גליקלעך

Yiddish expression of the day

"Zol men trinken af simkhes" = זאָל מען טרינקען אויף שׂימחות

"may you love to drink at parties" =

Chapter 3 - Matrix Algebra

Quickly Recall

A basis $\vec{v}_1 \ldots \vec{v}_n$ of $\mathbb{R}^n$ are a list of vectors that
1) are linearly independent
2) they span $\mathbb{R}^n$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \leftarrow \text{non-example}$$

Recall: The standard basis for $\mathbb{R}^n$ was the following

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \quad \vec{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$$

Remember that any $\vec{v}$ in $\mathbb{R}^n$ can be expressed as

$$\vec{V} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = v_1 \vec{e}_1 + v_2 \vec{e}_2 + v_3 \vec{e}_3 + \cdots + v_n \vec{e}_n$$

Recall 2: A subspace $W \subseteq \mathbb{R}^n$ is a subset that is

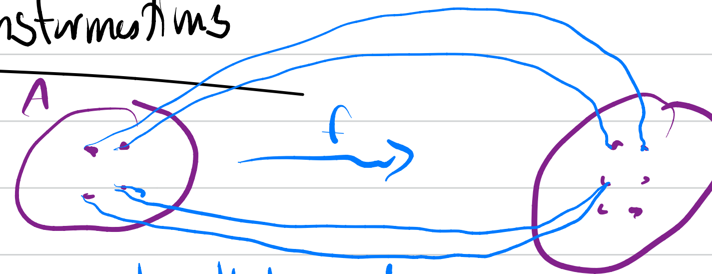
1) closed under addition
2) Closed under scalar multiplication

ex) $A = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_k \\ \downarrow & & \downarrow \end{pmatrix}_{n \times k}$ be this $n \times k$ matrix

Then the column space of A, denoted $\text{col}(A)$ is $\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$.

Chapter 3 - Matrices / Linear Transformations

Section 3.1 - Linear Transformations.



Def: A function $f: A \to B$ is a rule that sends for every input exactly one output

ii) A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is a function such that

1) $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$ for all vectors v, w

2) $T(c\vec{v}) = cT(\vec{v})$

ex) $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$

- Is T a linear transf?

1) Is $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$? Yes!

$$T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ x_1-y_1 \end{pmatrix}$$

$$T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_2+y_2 \\ x_2-y_2 \end{pmatrix}$$

$$T\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2+y_1+y_2 \\ x_1+x_2-(y_1+y_2) \end{pmatrix}$$

$$\Rightarrow T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1+y_1+x_2+y_2 \\ x_1-y_1+x_2-y_2 \end{pmatrix} = T\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix} \checkmark$$

2) Is $T(c\vec{v}) = cT(\vec{v})$? $T\begin{pmatrix} cx_1 \\ cy_1 \end{pmatrix} = \begin{pmatrix} cx_1+cy_1 \\ cx_1-cy_1 \end{pmatrix} = c\begin{pmatrix} x_1+y_1 \\ x_1-y_1 \end{pmatrix} = cT\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \checkmark$

ex) $T: \mathbb{R}^3 \to \mathbb{R}^2$ by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ Is this linear?

1) $$T\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \stackrel{?}{=} T\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix} \quad \checkmark$$

2) $$T\begin{pmatrix} cx_1 \\ cy_1 \\ cz_1 \end{pmatrix} \stackrel{?}{=} cT\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$\begin{pmatrix} cx_1 \\ cy_1 \end{pmatrix} = c\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \checkmark$$

ex3) $T: \mathbb{R}^2 \to \mathbb{R}$ by $T\begin{pmatrix} x \\ y \end{pmatrix} = x + y + 2$

$$T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \overset{?}{=} T\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$$

$$x_1 + y_1 + 2 \; + \; x_2 + y_2 + 2 \neq x_1 + x_2 + y_1 + y_2 + 2$$

Not a linear transformation!

Note $T\begin{pmatrix} cx \\ cy \end{pmatrix} \neq cT\begin{pmatrix} x \\ y \end{pmatrix}$

ex) $T: \mathbb{R}^4 \to \mathbb{R}^2$ by $T\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} xy \\ zw \end{pmatrix}$

Q: Is this linear?

$$T\begin{pmatrix} cx \\ cy \\ cz \\ cw \end{pmatrix} \qquad\qquad c\,T\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

$$\begin{pmatrix} c^2xy \\ c^2zw \end{pmatrix} \neq c\begin{pmatrix} xy \\ zw \end{pmatrix}$$

No!

Practice: Show $T\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix} \neq T\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \\ w_1+w_2 \end{pmatrix}$

Note: If T is a linear transformation
then $T\left(\vec{x}_1 + \cdots + \vec{x}_n\right) = T(\vec{x}_1) + T(\vec{x}_2) + \cdots + T(\vec{x}_n)$

Now let $\vec{e}_1, \ldots, \vec{e}_n$ be the standard basis of $\mathbb{R}^n$

Then $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n$

Then $T(\vec{v}) = T\left(v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n\right)$

$= T(v_1\vec{e}_1) + T(v_2\vec{e}_2) + \cdots + T(v_n\vec{e}_n)$

$= \underline{v_1 T(\vec{e}_1) + v_2 T(\vec{e}_2) + \cdots + v_n T(\vec{e}_n)}$

That is: A linear transf is uniquely defined by what it "does" to the standard basis!

ex) Suppose $T: \mathbb{R}^3 \to \mathbb{R}^2$ and

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}, \quad T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\6\end{pmatrix}, \quad T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-3\\1\end{pmatrix}$$

Q: What is $T\begin{pmatrix}2\\3\\4\end{pmatrix}$

$$\begin{pmatrix}2\\3\\4\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\\0\end{pmatrix} + 4\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$T\begin{pmatrix}2\\3\\4\end{pmatrix} = T\left(2\vec{e}_1 + 3\vec{e}_2 + 4\vec{e}_3\right) = T(2e_1) + T(3e_2) + T(4e_3)$$

$$= 2T(e_1) + 3T(e_2) + 4T(e_3)$$

$$T\begin{pmatrix}2\\3\\4\end{pmatrix} = 2\begin{pmatrix}2\\3\end{pmatrix} + 3\begin{pmatrix}1\\6\end{pmatrix} + 4\begin{pmatrix}-2\\1\end{pmatrix}$$

$$= \begin{pmatrix}4 + 3 - 8\\6 + 18 + 4\end{pmatrix} = \begin{pmatrix}-1\\28\end{pmatrix}$$

Def: Again let $\vec{e}_1, \ldots, \vec{e}_n$ be the standard basis for $\mathbb{R}^n$
and

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

The standard matrix associated to T is the $m \times n$ matrix

$$A_T = \begin{pmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(e_n) \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}_{m \times n}$$

ex) i) $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$ $\quad (T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2)$

- $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\longrightarrow A_T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

ii) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\qquad T: \mathbb{R}^3 \to \mathbb{R}^2$

$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\qquad T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$A_T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}_{2\times 3}$$

iii) Again consider the transf T where I just told you what T does to a basis

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}, \quad T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\6\end{pmatrix}, \quad T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-2\\1\end{pmatrix}$$

$$\longrightarrow A_T = \begin{pmatrix} 2 & 1 & -2 \\ 3 & 6 & 1 \end{pmatrix}_{2\times 3}$$

Def: Let $\vec{x}$ be a vector in $\mathbb{R}^n$, $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be linear $\longrightarrow A_T = \begin{pmatrix} T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ \downarrow & & \downarrow \end{pmatrix}_{m\times n}$

$$A_T \vec{x} \overset{def}{=} x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \cdots + x_n T(\vec{e}_n)$$

ex) Let A_T be as in example 3

Q: What is $A_T \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \overset{\text{def}}{=} 2\begin{pmatrix} 2 \\ 3 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 6 \end{pmatrix} + 4\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 4 \\ 6 \end{pmatrix} + \begin{pmatrix} 3 \\ 18 \end{pmatrix} + \begin{pmatrix} -8 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 28 \end{pmatrix}$$

$$= T\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \quad !!$$

1) Defining matrix mult with a vector this way "works" because for any vector $\vec{v}$

$$T(\vec{v}) = A_T \vec{v}$$

2) Consider the following system of equations

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ x_1 - 2x_2 + 3x_3 &= 8 \\ x_1 + 6x_2 - x_3 &= 1 \end{aligned} \quad \xleftrightarrow{\text{vector eqn}} \quad \begin{pmatrix} x_1 + 2x_2 - x_3 \\ x_1 - 2x_2 + 3x_3 \\ x_1 + 6x_2 - x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix}$$

$$\underbrace{x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -2 \\ 6 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}}_{} = \begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix}$$

$$||$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -2 & 3 \\ 1 & 6 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix}$$

$A\vec{x} = \vec{b}$!!

- That is, solving the linear system A is the same as finding some $\vec{x}$ such $A\vec{x} = \vec{b}$ has a solution

We have the following. The 3 ideas are equivalent

- A linear system with coefficient matrix $(A \,\vdots\, \vec{b})$ is consistent

$\Updownarrow$ (last week)

the vector $\vec{b}$ is in span (collumns of A) = col (A)

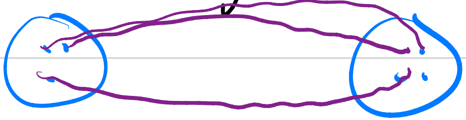
$\Updownarrow$ (from today)

As saying the matrix eqn $A\vec{x} = \vec{b}$ has a solution

That is, this equivalence tells us that solving systems of equations is really the same thing as understanding $T(\vec{x})$

Def: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformation.

1) The range of T $\quad R(T) = \{ \vec{w} \text{ in } \mathbb{R}^m : \text{there is some } \vec{v} \text{ in } \mathbb{R}^n \text{ with } T(\vec{v}) = \vec{w} \}$



2) The null space of the associated standard matrix A_T

$$\text{null}(A) = \{ \vec{x} \text{ in } \mathbb{R}^n : A\vec{x} = \vec{0} \}$$ conceptual

- by the above connection the collection of all vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$ is the solution set to the homogeneous system of equations $(A \,\vdots\, \vec{0})$

ex) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 6 \end{pmatrix}$ Find null(A).

Solve $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 0 \\ 3 & 5 & 6 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -1 & -3 & 0 \end{array}\right)$

$\xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)$

$\Rightarrow$ null $A = \{\vec{0}\}$

ex) $A = \begin{pmatrix} 1 & -2 & 3 \\ -1 & 4 & -6 \end{pmatrix}$ Find null(A)

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ -1 & 4 & -6 & 0 \end{array}\right) \xrightarrow{R_2 \to R_2 + R_1} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 2 & -3 & 0 \end{array}\right)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3r + 2\left(3/2\, r\right) \\ 3/2\, r \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 3/2\, r \\ r \end{pmatrix} = \boxed{\underset{\text{null}(A)}{\underset{\shortparallel}{\text{Span}\begin{pmatrix} 0 \\ 3/2 \\ 1 \end{pmatrix}}}}$$

Check: $\begin{pmatrix} 1 & -2 & 3 \\ -1 & 4 & -6 \end{pmatrix}\begin{pmatrix} 0 \\ 3/2 \\ 1 \end{pmatrix} = 0\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3/2\begin{pmatrix} -2 \\ 4 \end{pmatrix} + 1\begin{pmatrix} 3 \\ -6 \end{pmatrix}$

$$= \begin{pmatrix} -3 + 3 \\ 6 - 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \checkmark$$

In Summary: The following are equivalent

1) the columns of A are LI) last week
2) there's a pivot in each collumn
3) null$(A) = \{\vec{0}\}$) new characterization.

More generally

We saw that the linear system
$(A \mid \vec{b})$ has an answer precisely when
$A\vec{x}=\vec{b}$ has a solution.

Ie, a vector $\vec{b}$ is a LC of the collumns of matrix A
precisly when $A\vec{x}=\vec{b}$ for some $\vec{x}$ in $\mathbb{R}^n$

$$\begin{gathered} A\vec{x} \\ \shortparallel \\ T(\vec{x}) \end{gathered}$$

Hence, the following are equivalent

1) $[A \mid \vec{b}]$ is consistent for any vector $\vec{b}$
2) the columns of A span $\mathbb{R}^n$
3) A has a pivot in every row

old

4) there is an $\vec{x}$ such that $A\vec{x} = \vec{b}$ for any $\vec{b}$

new

ex) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$

Is the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ in the $R(T)$?

Ie does $T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\quad \shortparallel$

$$\underbrace{\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}}_{A_T}\begin{pmatrix}x_1\\ y_1\end{pmatrix}=\begin{pmatrix}2\\ 3\end{pmatrix}$$

$$\Longrightarrow \left(\begin{array}{cc|c}1 & 1 & 2\\ 1 & -1 & 3\end{array}\right) \xrightarrow{R_2\to R_2-R_1} \left(\begin{array}{cc|c}1 & 1 & 2\\ 0 & -2 & 1\end{array}\right)$$

$y_1=-1/2$ $\quad$ $x_1-1/2=2$ $\quad$ so $x_1=5/2$

$$T\begin{pmatrix}5/2\\ -1/2\end{pmatrix}=\begin{pmatrix}5/2-1/2\\ 5/2+1/2\end{pmatrix}=\begin{pmatrix}2\\ 3\end{pmatrix} \quad \checkmark$$